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Displacement Function of Shear-Flexible Orthotropic Plates on Elastic Foundation

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Synopsis

This paper derives the displacement function for the orthotropic plates resting on the Winkler's elastic foundation by using the third-order shear deformable plate theory of Reddy and the differential operator method. For the case of transversely isotropic and isotropic plates, the eighth-order differential equation related to the displacement function of the orthotropic plate is further resolved into sixth- and second-order differential equations, so called the interior and edge-zone equations, respectively.

Key Words: *differential operator method, displacement function, elastic (Winkler) foundation, Reddy plate, shear deformation, third-order plate theory*

1. Introduction

In the previous papers [1,2], the authors presented the displacement functions for solving the static and dynamic bending problems of orthotropic Reddy plates without elastic foundation, by use of the differential operator method [3]. Reddy's third-order shear deformable plate theory [4] is based on the displacement field expressed by the deflection and two rotations of a plate normal section. The displacement field accommodates quadratic variation of transverse shear strains (and hence stresses) and vanishing of transverse shear stresses on the top and bottom surfaces of the plate. Thus, unlike the first-order shear deformable plate theory such as Mindlin's plate theory [5], it does not require the use of a shear coefficient.

The objective of this paper is to extend the differential operator method to the orthotropic Reddy plates resting on elastic foundations obeying the Winkler's hypothesis. The coupled governing equations of the plate are transformed into the uncoupled eighth-order differential equation defined by a potential function (displacement function). For the case of transversely isotropic and isotropic plates, this eighth-order differential equation is further resolved into two independent equations, i.e., sixth- and second-order differential equations so called the interior equation and the edge-zone equation, respectively.

2. Governing Equations

The third-order shear deformable plate theory of Reddy [4] for bending problems is based on the following displacement field:

$$
\begin{aligned}
u_x(x,y,z) &= z\left[\psi_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\psi_x + \frac{\partial w}{\partial x}\right)\right],\\
u_y(x,y,z) &= z\left[\psi_y - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\psi_y + \frac{\partial w}{\partial y}\right)\right],\\
u_z(x,y,z) &= w(x,y), \qquad (1)
\end{aligned}
$$

where u_x, u_y and u_z are the components of displacement at a point (x,y,z) in x, y and z coordinates directions, respectively; w is the deflection; $\psi_x(x,y)$ and $\psi_y(x,y)$ are the rotations of a line element initially normal to the plate middle plane about the $y-$ and $x-$ axes, respectively; and h is the plate thickness.

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The principle of virtual displacements with the assumed displacement field of Eq. (1) and constitutive equations of the orthotropic plate may be used to derive the governing differential equations of Reddy plates resting on elastic foundation in the following matrix form:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} w \\ \psi_x \\ \psi_y \end{bmatrix} = \begin{bmatrix} -q \\ 0 \\ 0 \end{bmatrix}. \tag{2}$$

In the above, $q(x, y)$ is the intensity of lateral load, and L_{ij} are linear differential operators given by

$$L_{11} = a_1 \frac{\partial^4}{\partial x^4} + a_2 \frac{\partial^4}{\partial x^2 \partial y^2} + a_3 \frac{\partial^4}{\partial y^4} + a_4 \frac{\partial^2}{\partial x^2} + a_5 \frac{\partial^2}{\partial y^2} + k, \quad L_{12} = \frac{\partial}{\partial x}\left(a_6 \frac{\partial^2}{\partial x^2} + a_7 \frac{\partial^2}{\partial y^2} + a_4 \right),$$

$$L_{13} = \frac{\partial}{\partial y}\left(a_7 \frac{\partial^2}{\partial x^2} + a_8 \frac{\partial^2}{\partial y^2} + a_5 \right), \quad L_{21} = -L_{12}, \quad L_{22} = a_9 \frac{\partial^2}{\partial x^2} + a_{10} \frac{\partial^2}{\partial y^2} - a_4, \quad L_{23} = a_{11} \frac{\partial^2}{\partial x \partial y},$$

$$L_{31} = -L_{13}, \quad L_{32} = L_{23}, \quad L_{33} = a_{10} \frac{\partial^2}{\partial x^2} + a_{12} \frac{\partial^2}{\partial y^2} - a_5, \tag{3}$$

where k is the modulus of the elastic foundation, and a_i are constant coefficients related to the rigidities D_{ij} of the orthotropic plate as

$$a_1 = -\frac{1}{21} D_{11}, \quad a_2 = -\frac{2}{21}(D_{12} + 2D_{66}), \quad a_3 = -\frac{1}{21} D_{22}, \quad a_1 = \frac{32}{5h^2} D_{55},$$

$$a_5 = \frac{32}{5h^2} D_{44}, \quad a_6 = \frac{16}{105} D_{11}, \quad a_7 = \frac{16}{105}(D_{12} + 2D_{66}), \quad a_8 = \frac{16}{105} D_{22},$$

$$a_9 = \frac{68}{105} D_{11}, \quad a_{10} = \frac{68}{105} D_{66}, \quad a_{11} = \frac{68}{105}(D_{12} + D_{66}), \quad a_{12} = \frac{68}{105} D_{22}, \tag{4}$$

in which the plate rigidities D_{ij} are expressed by Young's moduli E_x, E_y, shear moduli G_{xy}, G_{yz}, G_{xz}, and Poisson's ratios ν_{xy}, ν_{yx}, as follows:

$$D_{11} = \frac{E_x h^3}{12(1 - \nu_{xy}\nu_{yx})}, \quad D_{22} = \frac{E_y h^3}{12(1 - \nu_{xy}\nu_{yx})}, \quad D_{12} = \frac{\nu_{xy} E_y h^3}{12(1 - \nu_{xy}\nu_{yx})},$$

$$D_{66} = \frac{G_{xy} h^3}{12(1 - \nu_{xy}\nu_{yx})}, \quad D_{44} = \frac{G_{yz} h^3}{12}, \quad D_{55} = \frac{G_{xz} h^3}{12}. \tag{5}$$

3. Solution Procedure

The governing differential equation (2) is a system of the coupled partial differential equations with constant coefficients, and thus the differential operator method (for example, Heki and Habara [3]) may be used to obtain the solutions w, ψ_x and ψ_y, which are given by

$$w = M_{11}(\phi), \quad \psi_x = M_{12}(\phi), \quad \psi_y = M_{13}(\phi), \tag{6}$$

where M_{1j} ($j = 1, 2, 3$) are cofactors to the elements L_{ij} of the determinant (L_{ij}) of Eq. (2):

$$M_{11} = L_{22}L_{33} - L_{23}L_{32}, \quad M_{12} = L_{23}L_{31} - L_{21}L_{33}, \quad M_{13} = L_{21}L_{32} - L_{22}L_{31}, \tag{7}$$

and the potential function ϕ should satisfy the following non-homogeneous differential equation:

$$\det(L_{ij})\phi = -q. \tag{8}$$

Substitution of Eq. (3) into Eq. (7) gives the explicit expressions for M_{11}, M_{12}, M_{13}, and hence the explicit expressions for w, ψ_x and ψ_y are also determined from Eq. (6) with the resulting expressions of M_{11}, M_{12}, M_{13} in the following forms:

$$w = \left\langle \frac{68^2}{105^2} \left\{ D_{11}D_{66} \frac{\partial^4}{\partial x^4} + [D_{11}D_{22} - D_{12}(D_{12} + 2D_{66})] \frac{\partial^4}{\partial x^2 \partial y^2} + D_{22}D_{66} \frac{\partial^4}{\partial y^4} \right\} \right.$$

$$-\frac{2{,}176}{525h^2}\left[(D_{11}D_{44}+D_{55}D_{66})\frac{\partial^2}{\partial x^2}+(D_{22}D_{55}+D_{44}D_{66})\frac{\partial^2}{\partial y^2}\right]+\left(\frac{32}{5h^2}\right)^2 D_{44}D_{55}\right\rangle\phi,$$

$$\psi_x=\frac{\partial}{\partial x}\left\langle\frac{1{,}088}{105^2}\left\{D_{11}D_{66}\frac{\partial^4}{\partial x^4}+[D_{11}D_{22}-D_{12}(D_{12}+2D_{66})]\frac{\partial^4}{\partial x^2\partial y^2}+D_{22}D_{66}\frac{\partial^4}{\partial y^4}\right\}\right.$$

$$\left.+\frac{128}{525h^2}\left\{[17D_{55}D_{66}-4D_{11}D_{44}]\frac{\partial^2}{\partial x^2}+[17D_{22}D_{55}-D_{44}(21D_{12}+25D_{66})]\frac{\partial^2}{\partial y^2}\right\}-\left(\frac{32}{5h^2}\right)^2 D_{44}D_{55}\right\rangle\phi,$$

$$\psi_y=\frac{\partial}{\partial y}\left\langle\frac{1{,}088}{105^2}\left\{D_{11}D_{66}\frac{\partial^4}{\partial x^4}+[D_{11}D_{22}-D_{12}(D_{12}+2D_{66})]\frac{\partial^4}{\partial x^2\partial y^2}+D_{22}D_{66}\frac{\partial^4}{\partial y^4}\right\}\right.$$

$$\left.+\frac{128}{525h^2}\left\{[17D_{11}D_{44}-D_{55}(21D_{12}+25D_{66})]\frac{\partial^2}{\partial x^2}+[17D_{44}D_{66}-4D_{22}D_{55}]\frac{\partial^2}{\partial y^2}\right\}-\left(\frac{32}{5h^2}\right)^2 D_{44}D_{55}\right\rangle\phi. \tag{9}$$

Further, $\det(L_{ij})$ of Eq. (8) is expanded to find the following fundamental equation of eighth-order for the potential function ϕ:

$$L_1(\phi)-kL_2(\phi)=-q, \tag{10}$$

where L_1 and L_2 are the differential operators defined as

$$L_1=b_1\frac{\partial^8}{\partial x^8}+b_2\frac{\partial^8}{\partial x^6\partial y^2}+b_3\frac{\partial^8}{\partial x^4\partial y^4}+b_4\frac{\partial^8}{\partial x^2\partial y^6}+b_5\frac{\partial^8}{\partial y^8}$$

$$+b_6\frac{\partial^6}{\partial x^6}+b_7\frac{\partial^6}{\partial x^4\partial y^2}+b_8\frac{\partial^6}{\partial x^2\partial y^4}+b_9\frac{\partial^6}{\partial y^6}+b_{10}\frac{\partial^4}{\partial x^4}+b_{11}\frac{\partial^4}{\partial x^2\partial y^2}+b_{12}\frac{\partial^4}{\partial y^4},$$

$$L_2=c_1\frac{\partial^4}{\partial x^4}+c_2\frac{\partial^4}{\partial x^2\partial y^2}+c_3\frac{\partial^4}{\partial y^4}+c_4\frac{\partial^2}{\partial x^2}+c_5\frac{\partial^2}{\partial y^2}+c_6, \tag{11}$$

in which constant coefficients $b_i\,(i=1-12)$ and $c_i\,(i=1-6)$ are given as follows:

$$b_1=-\frac{5{,}712}{105^3}D_{11}^2D_{66},\quad b_2=-\frac{5{,}712}{105^3}D_{11}(D_{11}D_{22}-D_{12}^2+4D_{66}^2),$$

$$b_3=\frac{11{,}424}{105^3}[D_{12}(D_{12}+2D_{66})^2-D_{11}D_{22}(D_{12}+3D_{66})],\quad b_4=-\frac{5{,}712}{105^3}D_{22}(D_{11}D_{22}-D_{12}^2+4D_{66}^2),$$

$$b_5=-\frac{5{,}712}{105^3}D_{22}^2D_{66},\quad b_6=\frac{128}{2{,}625h^2}D_{11}(D_{11}D_{44}+85D_{55}D_{66}),$$

$$b_7=\frac{128}{2{,}625h^2}[D_{11}D_{44}(2D_{12}+89D_{66})+D_{55}(85D_{11}D_{22}-84D_{12}^2-166D_{12}D_{66}+4D_{66}^2)],$$

$$b_8=\frac{128}{2{,}625h^2}[D_{22}D_{55}(2D_{12}+89D_{66})+D_{44}(85D_{11}D_{22}-84D_{12}^2-166D_{12}D_{66}+4D_{66}^2)],$$

$$b_9=\frac{128}{2{,}625h^2}D_{22}(D_{22}D_{55}+85D_{44}D_{66}),\quad b_{10}=-\left(\frac{32}{5h^2}\right)^2D_{11}D_{44}D_{55},$$

$$b_{11}=-2\left(\frac{32}{5h^2}\right)^2D_{44}D_{55}(D_{12}+2D_{66}),\quad b_{12}=-\left(\frac{32}{5h^2}\right)^2D_{22}D_{44}D_{55}, \tag{12}$$

and

$$c_1=\frac{68^2}{105^2}D_{11}D_{66},\quad c_2=\frac{68^2}{105^2}[D_{11}D_{12}-D_{12}(D_{12}+2D_{66})],\quad c_3=\frac{68^2}{105^2}D_{22}D_{66},$$

$$c_4=-\frac{2{,}176}{525h^2}(D_{11}D_{44}+D_{55}D_{66}),\quad c_5=-\frac{2{,}176}{525h^2}(D_{22}D_{55}+D_{44}D_{66}),\quad c_6=\left(\frac{32}{5h^2}\right)^2D_{44}D_{55}. \tag{13}$$

When the term depending on the modulus of the elastic foundation k from Eq. (10), the solutions w, ψ_x and ψ_y and the potential function ϕ are identical with those of bending problems without the foundation, which have already been obtained by the authors [1].

4. Transversely Isotropic and Isotropic Plates

4.1 Transversely isotropic plate

For a transversely isotropic plate with the plane of isotropy parallel to the middle plane of the plate, the material constants appearing in Eq. (5) can be written as $E_x = E_y = E$, $G_{xz} = G_{yz} = G_z$, $\nu_{xy} = \nu_{yz} = \nu$, and $G_{xy} = G$, in which $E, G[= E/2(1+\nu)]$, and ν, respectively, are Young's modulus, shear modulus and Poisson's ratio for the plane of isotropy. Therefore, the plate rigidities D_{ij} and coefficients a_i for the case of orthotropic plate in Eqs. (5) and (4), respectively, are given by

$$D_{11} = D_{22} = D,\quad D_{12} = \nu D,\quad D_{66} = \frac{1-\nu}{2}D,\quad D_{44} = D_{55} = D_z, \tag{14}$$

and

$$a_1 = a_3 = -\frac{1}{21}D,\quad a_2 = 2a_1,\quad a_4 = a_5 = \frac{32}{5h^2}D_z,\quad a_6 = a_7 = a_8 = \frac{16}{105}D,$$
$$a_9 = a_{12} = \frac{68}{105}D,\quad a_{10} = \frac{1-\nu}{2}a_9,\quad a_{11} = a_9 - a_{10} = \frac{1+\nu}{2}a_9, \tag{15}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)},\quad D_z = \frac{G_z h^3}{12}. \tag{16}$$

Further, the differential operators L_{ij} and L_i of Eqs. (3) and (11), respectively, are similarly reduced to

$$L_{11} = a_1\nabla^2\nabla^2 + a_4\nabla^2 + k,\quad L_{12} = \frac{\partial}{\partial x}(a_6\nabla^2 + a_4),\quad L_{13} = \frac{\partial}{\partial y}(a_6\nabla^2 + a_4),$$
$$L_{21} = -L_{12},\quad L_{22} = a_9\frac{\partial^2}{\partial x^2} + a_{10}\frac{\partial^2}{\partial y^2} - a_4,\quad L_{23} = (a_9 - a_{10})\frac{\partial^2}{\partial x \partial y},$$
$$L_{31} = -L_{13},\quad L_{32} = L_{23},\quad L_{33} = a_{10}\frac{\partial^2}{\partial x^2} + a_9\frac{\partial^2}{\partial y^2} - a_4, \tag{17}$$

and

$$L_1 = \nabla^2\nabla^2(a_{10}\nabla^2 - a_4)[(a_1a_9 + a_6^2)\nabla^2 - a_4(a_1 - 2a_6 - a_9)],$$
$$L_2 = (a_{10}\nabla^2 - a_4)(a_9\nabla^2 - a_4), \tag{18}$$

where ∇^2 is the Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{19}$$

Subsititution of Eq. (17) into Eq. (7) gives the explicit expressions M_{11}, M_{12}, M_{13}, and then, from Eq. (6), the solutions w, ψ_x and ψ_y for the case of transversely isotropic plate are expressed by

$$w = (a_{10}\nabla^2 - a_4)(a_9\nabla^2 - a_4)\phi,$$
$$\psi_x = \frac{\partial}{\partial x}(a_{10}\nabla^2 - a_4)(a_6\nabla^2 + a_4)\phi,$$
$$\psi_y = \frac{\partial}{\partial y}(a_{10}\nabla^2 - a_4)(a_6\nabla^2 + a_4)\phi, \tag{20}$$

where the potential function ϕ must satisfy the following non-homogeneous differential equation obtained from Eq. (10) inconjunction with Eq. (18):

$$(a_{10}\nabla^2 - a_4)\{\nabla^2\nabla^2[(a_1a_9 + a_6^2)\nabla^2 - a_4(a_1 - 2a_6 - a_9)] - k(a_9\nabla^2 - a_4)\}\phi = -q. \tag{21}$$

Equations (20) and (21) contain the common differential operator, $a_{10}\nabla^2 - a_4$, and thus this common operator must be factored out to obtain the complete solutions. After factoring out this common operator, substituting a_i of Eq. (15) into the resulting equations, and then introducing new potential function Φ as

$$\Phi = -\frac{a_1 a_9 + a_6^2}{D}\phi = \frac{4D}{525}\phi, \tag{22}$$

the solutions w, ψ_x and ψ_y are determined as

$$w = 85\left(\nabla^2 - \frac{168D_z}{17Dh^2}\right)\Phi,$$

$$\psi_x = 20\frac{\partial}{\partial x}\left(\nabla^2 + \frac{42D_z}{Dh^2}\right)\Phi,$$

$$\psi_y = 20\frac{\partial}{\partial y}\left(\nabla^2 + \frac{42D_z}{Dh^2}\right)\Phi, \tag{23}$$

where the potential function Φ must satisfy the following sixth-order equation, so called the interior equation:

$$\left[\nabla^2\nabla^2\left(\nabla^2 - \frac{840D_z}{Dh^2}\right) + 85\frac{k}{D}\left(\nabla^2 - \frac{168D_z}{17Dh^2}\right)\right]\Phi = \frac{q}{D}. \tag{24}$$

Reddy's plate equation is essentially the eighth-order differential equation as shown in Eq. (10). Thus to obtain another type of solutions which satisfy the second-order differential equation, introduce a potential function Ψ as

$$w = 0, \quad \psi_x = \frac{\partial \Psi}{\partial y}, \quad \psi_y = -\frac{\partial \Psi}{\partial x}. \tag{25}$$

Substituting these into the governing equation of Eq. (2) in conjunction with differential operators L_{ij} of Eq. (17), the first equation with $q = 0$ is automatically satisfied. And then, from the second and third equations of Eq. (2), the following second-order differential equation, so called the edge-zone equation, is obtained:

$$\left[\nabla^2 - \frac{336D_z}{17(1-\nu)Dh^2}\right]\Psi = 0. \tag{26}$$

Therefore, the complete solutions for the transversely isotropic plate are given by two potential functions Φ and Ψ as

$$w = 85\left(\nabla^2 - \frac{168D_z}{17Dh^2}\right)\Phi,$$

$$\psi_x = 20\frac{\partial}{\partial x}\left(\nabla^2 + \frac{42D_z}{Dh^2}\right)\Phi + \frac{\partial \Psi}{\partial y},$$

$$\psi_y = 20\frac{\partial}{\partial y}\left(\nabla^2 + \frac{42D_z}{Dh^2}\right)\Phi - \frac{\partial \Psi}{\partial x}. \tag{27}$$

4.2 Isotropic plate

For the isotropic plate, from Eq. (16),

$$G_z = G = \frac{E}{2(1+\nu)}, \quad D_z = \frac{1-\nu}{2}D, \tag{28}$$

and thus the expressions w, ψ_x and ψ_y of Eq. (27) become

$$w = 85\left[\nabla^2 - \frac{84(1-\nu)}{17h^2}\right]\Phi,$$

$$\psi_x = 20\frac{\partial}{\partial x}\left[\nabla^2 + \frac{21(1-\nu)}{h^2}\right]\Phi + \frac{\partial \Psi}{\partial y},$$

$$\psi_y = 20\frac{\partial}{\partial y}\left[\nabla^2 + \frac{21(1-\nu)}{h^2}\right]\Phi - \frac{\partial \Psi}{\partial x}. \tag{29}$$

At the same time, two potential functions Φ and Ψ must satisfy the following interior and edge-zone equations from Eqs. (24) and (26), respectively

$$\left\{\nabla^2\nabla^2\left[\nabla^2 - \frac{420(1-\nu)}{h^2}\right] + 85\frac{k}{D}\left[\nabla^2 - \frac{84(1-\nu)}{17h^2}\right]\right\}\Phi = \frac{q}{D}, \tag{30}$$

$$\left[\nabla^2 - \frac{168}{17h^2}\right]\Psi = 0. \tag{31}$$

5. Concluding Remarks

The differential operator method is developed for a system of the coupled governing differential equations of the orthotropic plates resting on elastic foundations, which is based on the Reddy's third-order shear deformable plate theory. The governing differential equations are transformed into the uncoupled eighth-order differential equation with respect to one potential function. For the case of transversely isotropic and isotropic plates, this eighth-order differential equation is further resolved into two independent differential equations, i.e., second- and sixth-order differential equations, so called the interior and edge-zone equations, respectively.

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